

Monday, November 17 , Lecture 33: Green's functions

Last time...

• Solutions to heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) T = 0$$

$$T(x, 0) = f(x)$$

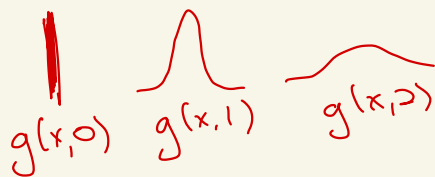
are given by

$$T(x, t) = g(x, t) * f(x)$$

$$= \int_{-\infty}^{\infty} \underbrace{G_t(x, y)}_{\text{heat kernel (looks like a Gaussian)}} f(y) dy$$

- At time $t=0$, $g(x, 0) = \delta$, so

$$T(x, 0) = (\delta * f)(x) = f(x).$$



• Derivatives and integrals of δ :

• $\delta_h^{(n)}(x)$ has property that for a test function f ,

$$\int_{-\infty}^{\infty} \delta_h^{(n)}(x) f(x) dx = (-1)^n f^{(n)}(h).$$

$$\int_{-\infty}^t \delta_h(x) = H_h(t) = \begin{cases} 0 & t < h \\ 1 & t > h \end{cases}$$

Motivation: Inverse operators.

Suppose we want to solve

$$A y = b$$

$\begin{matrix} n \times n & n \times 1 & 1 \times n \\ \left(\begin{matrix} & & \end{matrix} \right) & \left(\begin{matrix} \\ \end{matrix} \right) & \left(\begin{matrix} & & \end{matrix} \right) \end{matrix}$

for many different b . One approach is to find the inverse matrix A^{-1} :

$$y = A^{-1} A y = A^{-1} b.$$

How do we compute the inverse? Well, we can check

what it does on a basis. write $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + b_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

and call solutions $g_1, \dots, g_n$

$$g_1 = A^{-1} \delta_1, g_2 = A^{-1} \delta_2, \dots, g_n = A^{-1} \delta_n$$

Then

$$\begin{aligned} A^{-1} b &= A^{-1} (b_1 \delta_1 + b_2 \delta_2 + \dots + b_n \delta_n) \\ &= b_1 A^{-1} \delta_1 + b_2 A^{-1} \delta_2 + \dots + b_n A^{-1} \delta_n \\ &= b_1 g_1 + b_2 g_2 + \dots + b_n g_n \end{aligned}$$

or in other words, the solution to $A y = b$ is given by

$$x = \sum_{i=1}^n g_i b_i \quad \text{with each } g_i = A^{-1} \delta_i, \text{ or}$$
$$= G b \quad \left(\begin{matrix} | & | & \dots & | \\ g_1 & g_2 & \dots & g_n \\ | & | & \dots & | \end{matrix} \right) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

So $G = A^{-1}$
and columns g_i solve $A g_i = \delta_i$.

Moral: if we have solutions g_i to

$$Ag_i = \delta_i,$$

we can find solution for any $Ag = b$,
since

$$b = \sum_{i=1}^n b_i \delta_i$$

- One nice way to think about this decomposition:
Think of b as a function on $\{1, 2, \dots, n\}$

Then

$$b(i) = (\delta_0 * b)(i)$$

$$= \sum_j \delta_0(i-j) b(j)$$

$$\delta_0(i-j) = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$= b(i) \delta_i(i)$$

↑ on a ring, so

$n=0$

$n \cdot 0 \cdot 2$

$\cdot \cdot \cdot$

But now, let's upgrade this to linear differential equations:

$$ay'' + by' + cy = f$$

$$\underbrace{\left(a \frac{d^2}{dx^2} + b \frac{d}{dx} + c\right)}_A y = f \quad + \text{boundary conditions on } y.$$

$\uparrow$
 linear operator
 $\uparrow$
 $y, f: \mathbb{R} \rightarrow \mathbb{C}$

Now, we think of f as a forcing function.
We can break f into basic pieces by using

$$f(x') = \delta * f(x) = \int_{-\infty}^{\infty} \delta_{x'}(x) f(x) dx$$

← integral version
of $f = \sum_{i=1}^n \delta_i f_i$

Def: Green's function

Fix $x' \in \mathbb{R}$. The Green's function $G(x, x')$ is the solution to

$$(*) \quad A y(x) = \delta_{x'}(x) + \text{boundary conditions}$$

HW: Integral operators I_G

$$(I_G(f))(x) = \int_{-\infty}^{\infty} \underbrace{G(x, x')}_{\text{G is like an infinite matrix}} f(x') dx'$$

G is like an infinite matrix.

So this is like matrix multiplication

$$\begin{pmatrix} G \end{pmatrix}_{n \times n} \begin{pmatrix} f \end{pmatrix}_{n \times 1} = \begin{pmatrix} Gf \end{pmatrix}_{n \times 1}$$

$$\sum_{j=1}^n G_{ij} f_j = (Gf)_i$$

Payoff: if we know G , we can solve

since $Ay = f$

$$A(I_G f) = A \int_{-\infty}^{\infty} G(x, x') f(x') dx'$$

A only takes
derivatives $\frac{d}{dx}$
not x' .

$$\rightarrow = \int_{-\infty}^{\infty} A G(x, x') f(x') dx'$$

$$(*) = \int_{-\infty}^{\infty} \delta_{x'}(x) f(x') dx'$$

$$= f(x)$$

$$\Rightarrow I_G^{-1} = A!$$

In practice, Green's functions can be tricky to find.

They depend heavily on both A and boundary conditions.

Ex: Harmonic oscillator with forcing function $f(t)$

$$y'' + y = f$$

$$y(0) = 0$$

$$y(\pi/2) = 0$$

Find the Green's function.

Soln:

by definition,

$$\frac{d^2}{dt^2} G(t, t') + G(t, t') = \delta_t(t')$$

Boundary conditions:

$$G(0, t') = 0$$

$$G(\pi/2, t') = 0$$



• Let's try to solve this.

Step ①: find solutions on either side of $\delta_t(t')$.

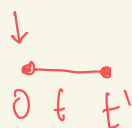
• If $0 < t < t' < \pi/2$

$$\frac{d^2}{dt^2} G(t, t') + G(t, t') = 0$$

$$+ G(0, t') = 0$$

$$\Rightarrow G(t, t') = A(t') \sin(t)$$

$$G(0, t') = 0$$



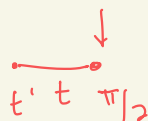
• If $0 < t' < t < \pi/2$

$$\frac{d^2}{dt^2} G(t, t') + G(t, t') = 0$$

$$+ G(\pi/2, t') = 0$$

$$\Rightarrow G(t, t') = B(t') \cos(t)$$

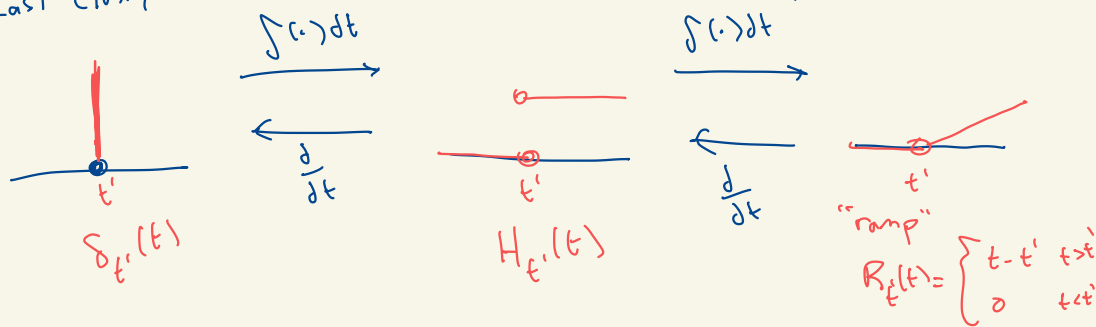
$$G(\pi/2, t') = 0$$



So... what happens at $t' = t$?

Step ②: Continuity of G

Last class, we saw how to take derivatives of $\delta_{t'}$ and



- Taking more derivatives of $\delta_{t'}$ gives even sharper discontinuities.

- Since eqn

$$\frac{d^2}{dt^2} G(t, t') + G(t, t') = \delta_{t'}(t)$$

$G(t, t')$ is smoother
 $\downarrow$ then $\frac{d^2}{dt^2} G(t, t')$

no higher derivatives
 $\delta_{t'}^{(n)}(t)$

So at worst, $G(t, t')$ has a discontinuity in 1st derivative.

$\Rightarrow G(t, t')$ looks like a ramp function,
 i.e. G is continuous.

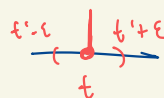
So we can equate the solutions $\overset{t=t'}{(*)}$ for $t < t'$ and $t > t'$:

$$A(t') \sin(t') = B(t') \cos(t').$$

We still need a bit more info to determine G .

Step ③: The derivative jump and gluing solutions

well... integrate both sides



$$\int_{t'-\varepsilon}^{t'+\varepsilon} \frac{d^2}{dt^2} G(t, t') + G(t, t') dt = \int_{t'-\varepsilon}^{t'+\varepsilon} \delta_{t'}(t) dt$$

$$= 1$$

$$\frac{dG}{dt}(t'+\varepsilon, t') - \frac{dG}{dt}(t'-\varepsilon, t') + \int_{t'-\varepsilon}^{t'+\varepsilon} G(t, t') dt$$

since G is continuous
and $\varepsilon \rightarrow 0$.

So:

$$\frac{dG}{dt}(t'+\varepsilon, t') - \frac{dG}{dt}(t'-\varepsilon, t') = 1$$

$t = t' + \varepsilon > t'$ $t = t' - \varepsilon < t'$

← $\frac{dG}{dt}$ is like
step function
~
This measures
the jump!

which means

$$-B(t') \sin(t'+\varepsilon) - A(t') \cos(t'-\varepsilon) = 1$$

Take $\varepsilon \rightarrow 0$. Together with (*), we have two eqns:

$$A(t') \sin(t') - B(t') \cos(t') = 0$$

$$-A(t') \cos(t') - B(t') \sin(t') = 1$$

Do some algebra $\Rightarrow A(t') = -\cos t'$

$$B(t') = -\sin t'$$

Finally:

$$G(t, t') = \begin{cases} -\cos t' \sin t & 0 < t < t' < \pi/2 \\ -\sin t' \cos t & 0 < t' < t < \pi/2 \end{cases}$$

Ex: if $f(t) = \csc(t)$, then soln to $y'' + y = f$ is

$$\begin{aligned} y(t) &= \int_0^{\pi/2} G(t, t') f(t') dt' \\ &= \int_0^t -\cos t \sin t' \csc t' dt' + \int_t^{\pi/2} -\sin t \cos t' \csc t' dt' \\ &= -t \cos t + \sin t \log(\sin t). \end{aligned}$$